Multiple risky securities valuation I.

Ilya, Gikhman

2008

Multiple risky securities valuation I.

Ilya Gikhman
6077 Ivy Woods Court
Mason OH 45040
Email: iljogik@yahoo.com
Phone: 513-573-9348

Abstract. In this paper we develop an approach to valuation of a multiple names security portfolio. The goal of the paper to present pricing and calculation of the risk characteristics of the corporate debt based on randomization of the historical data of portfolio assets. Our approach close but it does not coincide with the reduced form interpretation of the credit risk. Based on stochastic interpretation of the default it follows that the market price of a bond is a stochastic process. Therefore, a spot price of a corporate bond implies risk and the bond value shows how market weights the risk. We will show in details how default correlation within securities will affect the basket exposure.

JEL category: G13 Contingent Pricing

Key words. Credit derivatives, risky portfolio valuation, copula, perfect copula, CDS, CDO.

Introduction.

This paper was prepared in the middle of 2008 year. It was happened that global financial troubles almost eliminated the interest to this particular area of finance. Nevertheless, recently looking through the original draft it seemed to me that it might make sense to present this draft. It represents somewhat different approach to portfolio valuation.

We begin with some comments to benchmarks approach used for valuation of a portfolio of multiple correlated assets. The initial step was done in [6]. This paper provides a framework for deriving the default probability on a firm. It was assumed there that the asset value V(t) is a random process on a probability space { Ω, F, P } which follows

$$d A (t) = \mu A (t) d t + \sigma A (t) d W (t) \qquad (0.1)$$

The default event $D(T) \in F$ was assumed could occurred at the end of a period $[0, T]$. The probability of default was defined as

$$P\{A(T) < D(T)\} = \Phi(-d_1),$$

where $\Phi(\ )$ is the normal commutative distribution function and

$$d_1 = \frac{\ln \dfrac{A(0)}{D(T)} + (\mu - \dfrac{\sigma^2}{2})T}{\sigma \sqrt{T}}$$

Recall that the linear SDE of the type (0.1) was used for the approximation of the equity price. It insure that two or n stocks offer the same rate of return as the one stock. Here n is assumed a small number compare with the market volumes traded during the period $[0, T]$. Equation (0.1) should be changed if we deal with a size approximately equal or more than day trading. In particular, the pricing equation can be a nonlinear SDE. There exists an upper bound for the market volume of the stock. It is the stock open interest.

Next step in corporate pricing was made [7-9]. In these papers the portfolio of debt securities was studied. Let $A_i(t)$ be a security price at the date t and

$$dA_i(t) = \mu_i A_i(t)\,dt + \sigma_i A_i(t)\,dW_i(t)$$

where $W_i(t)$, $i = 1, \ldots, n$ are correlated Wiener processes $E\,W_i(t)\,W_j(t) = \rho t$, for $i \neq j$. The probability of default defined above for each debt was assumed to be equal and default could occurred only at the end of the period $[0, T]$. The asymptotic behavior of the portfolio's losses when n tends to infinity was presented in [8]. In [7] was used first a representation that has been served later as a prototype of copula's techniques.

Statement 1.

Let X_i be jointly normally distributed random variables with pair correlation ρ. Then

$$X_i = \sqrt{\rho}\,Y + \sqrt{1-\rho}\,Z_i \qquad (0.2)$$

Here Y and Z_i, $i = 1,2,\ldots n$ are mutually independent standard normal random variables. The first term on the right hand side was interpreted as the i-th company exposure to common factor Y (such as the state of economy) and the second term on the right (0.2) the company the company risk. Now variables Y and Z_i are called systematic and idiosyncratic terms correspondingly.

As far as this statement is a mathematical result we formulate it more accurately. Next idea of the proof suggested by Vasicek. One can check that representation (0.2) could be achieved by putting

$$Z_i = \frac{1}{\sqrt{1-\rho}} (X_i - Y\sqrt{\rho}) , \qquad Y = a \sum_{i=1}^{n} X_i + bU ,$$

where $a = \dfrac{\sqrt{\rho}}{1 + (n - 1)\rho}$, $b = \dfrac{\sqrt{1 - \rho}}{\sqrt{1 + (n-1)\rho}}$.

Thus, the in above statement one assumes that for a given set of n random variables X_i there exists a standard normal random variable $U = U_n$. Hence, the corrected statement can be reformulated as the following theorem.

Statement 2.

Let X_i be jointly normally distributed random variables with equal paired correlation ρ and let U be a standard normal random variable independent on X_i. Then the representation (0.2) holds.

The proof of the theorem is based on testing (0.2) given above formulas. This theorem was used in modeling asymptotic losses of a large portfolio [7-9].

Remark. The first obstacle in application of this statement relates to the random variable U. It is clear that this variable should be defined when the problem is set. Some authors tried to interpret the variable Y as a macroeconomic index or another macroeconomic parameter. We argue that it might make sense to assume that individual assets do not have significant impact on say DJI index. On the other hand if the index sufficiently changes it is difficult to expect that a particular set of assets will be independent on these index changes.

The next is a technical remark that relates to the risk neutral valuation. Merton and Vasicek [6,9] defined default in the 'real' world.

Following Black Scholes' option pricing which replaced real underlying security by its risk neutral counterpart by setting real security on the risk neutral world { Ω, F, Q } the price of credit instruments were neutralized by replacing their real drift coefficients by the risk free interest rate. This remark relates either to structural or reduced form approaches to the credit modeling. Some formal aspects of this problem were discussed in [3].

Multi-names portfolio valuation.

Let us now consider a basket of risky bonds. Following [2] consider a pricing model of a basket of risky bonds. We defined the market price of a risky bond at the date t with maturity T and recovery rate Δ which can defaults only at maturity as

$$\begin{aligned} R(t , T ; \omega) &= B(t , T) \{ [1 - \chi(\omega , D)] + \Delta \chi(\omega , D) \} = \\ &= B(t , T) \{ 1 - (1 - \Delta) \chi(\omega , D) \} \end{aligned} \qquad (1)$$

Here D denotes default event of the bond and Δ is assumed to be deterministic. From (1) it follows that

$$E\, R(t , T ; \omega) = B(t , T) [1 - (1 - \Delta) P(D)]$$

$$\mathrm{STDV}\,R(t,T;\omega) = B(t,T)(1-\Delta)\sqrt{P(D)(1-P(D))}$$

One can note that the market price given by (1) is a random function and therefore a particular spot price implies risk. This risk is associated with the event D which makes the price value (1) be bellow than the riskless price $B(t, T)$ at any moment t before maturity T. The risk free price can also be assumed stochastic though here we consider the case when B is deterministic.

There is a difference between (1) and reduced form pricing of the risky bond. We distinct the market price of the corporate bond given by (1) and the reduced form pricing. The market price at t is a random variable which values comes from the possibility of default. Note that (1) holds when recovery rate Δ is a random variable. We interpret the spot rate as equilibrium between demand and surplus that can be represented as a particular statistics of the random variable $R(t, T; \omega)$. Reduced form model deals with the price defined as the expected value of the $R(t, T; \omega)$. Thus the corporate bond price is took to the particular statistics in the reduced form model. We can interpret this reduction as the spot price. It makes sense when market is in equilibrium and demand and surplus does not remarkably change in time and therefore the expectation of the market price is a good estimate of the spot price.

Let $R_i(t, T_i; \omega)$ denote a risky bond price issued by i-company i = 1, 2,…, N and suppose that bonds might default only at maturity. Let $T_i \le T_{i+1}$, i = 1, 2,…, N, and denote $\chi_i(D_i) = \chi(\omega, T_i, D_i)$ the indicator function of the default scenario D_i occurred by assumption only at T_i. Denote $P_{ij} = P(D_i \cap D_j) = E\chi_i\chi_j$ the joint default distribution of the i-th and j-th bonds. Let us calculate the joint probability of default P_{ij} of two bonds. From (1) it follows that

$$\theta_i(t, T_i; \omega) \stackrel{def}{=} 1 - \frac{R_i(t, T_i; \omega)}{B(t, T_i)} = (1-\Delta_i)\chi_i \qquad (2)$$

Then

$$P_{ij} = E\chi_i\chi_j = E\frac{\theta_i\theta_j}{(1-\Delta_i)(1-\Delta_j)} \qquad (3)$$

This formula represents the joint distribution of the credit events given that default occurs at maturity only. Having joint distributions of two defaults it is easy to present the conditional probability $P_{j|i} = P(D_j \mid D_i)$ of default. Indeed, if $i < j$ then by the definition of the conditional probability we have $P_{j|i} = P_{ij} / P_i$ where $P_i = P(D_i)$ is the default probability of the i-th bond [2]

$$P_i = \frac{1}{1-\Delta_i}\left[1 - \frac{ER(t, T_i; \omega)}{B(t, T_i)}\right] \qquad (4)$$

The formula for multivariate defaults can be the represented in the form

$$P_{12\ldots k} = P\{\bigcap_{j=1}^{k} D_j\} = E\prod_{j=1}^{k}\chi_j = E\prod_{j=1}^{k}\frac{\theta_j}{1-\Delta_j}$$

These formulas present the basic correlation structure of the join defaults of the risky bonds when bonds default at maturity. This approach can also be used for calculations more complex multivariate distributions. The probability that a set of bonds marked by indexes J default at their maturities and a group of bonds specified by the set of indexes Λ would not default can be expressed by the formula

$$P\{\bigcap_{j\in J} D_j \cap \bigcap_{\lambda\in\Lambda}\overline{D}_\lambda\} = E\prod_{j\in J}\chi_j\prod_{\lambda\in\Lambda}\overline{\chi}_\lambda = E\prod_{j\in J}\frac{\theta_j}{1-\Delta_j}\prod_{\lambda\in\Lambda}[1-\frac{\theta_\lambda}{1-\Delta_\lambda}] \qquad (5)$$

Remark. The primary distinction between using probability formulas and statistical modeling is that in probability formulas we usually assume that all distributions and its parameters are known. For a statistical modeling historical data or other observations are used as a source of samples in order to perform tests and estimates of the model parameters. For some cases it might be difficult to produce statistical inference regarding unknown parameters of the model. In this case one may apply a mixed approach in which statistical approach could be combined with theoretical assumptions. Note that based on mathematical statistics the observations upon prices would straightforward lead to the statistical interpretation of the recovery rates [2].

Now consider pricing dynamics of the portfolio. Let T_i , i = 1, 2,…, N be a non decreasing sequence of maturities and suppose that default of the i-th bond might occur only at T_i. The conditional expectation of the corporate bond price R_i, $i > 1$ given default of the first bond at T_1 is

$$E\{R_i(T_1+0, T_i;\omega)\,|\,D_1\} = B(T_1, T_i)[1-(1-\Delta_i)E\{\chi_i|D_1\}] =$$

$$= B(T_1, T_i)[1-(1-\Delta_i)\frac{P_{1i}}{P_1}]$$

where probabilities P_{1j} and P_i are defined in (3, 3'). Bearing in mind that indicator of the no-default of the bond R_1 at T_1 is equal to $1-\chi_1$ it follows that the value of the risky bonds $R_i(t, T_i;\omega)$, i = 2, 3, … , L immediately after T_1 conditioning on no-default at the date T_1 is equal to

$$E\{R_i(T_1+0, T_i;\omega)\,|\,\overline{D}_1\} = B(T_1, T_i)[1-\frac{(1-\Delta_i)}{1-P_1}(P_{1i}-P_1)]$$

Let $n_i \geq 0$ be a number of shares of the corporate bond R_i in the portfolio Π and denote

$$\Pi(t,\overline{\mathbf{T}};\omega)=\sum_{i=1}^{L} n_i R_i(t,T_i;\omega),\ \overline{\mathbf{T}}=(T_1,T_2,\ldots,T_L)$$

Let k be a number, J a set of the indexes do not exceed k – 1 , and $\Lambda = \{1, 2, \ldots k-1\} \setminus J$. Then the value of the portfolio at t conditioning on defaults at J and no default at Λ is equal to

$$E\{\Pi(t,\overline{\mathbf{T}};\omega)\mid \bigcap_{\substack{j\in J\\ \lambda\in\Lambda}}(D_j\cap\overline{D}_\lambda)\} = \sum_{i=1}^{L} E\{R_i(t,T_i;\omega)\mid \bigcap_{\substack{j\in J\\ \lambda\in\Lambda}}(D_j\cap\overline{D}_\lambda)\} =$$

$$= \sum_{i=1}^{L} E\{B_i(t,T_i)[1-(1-\Delta_i)\chi(\omega,D_i)]\mid \bigcap_{\substack{j\in J\\ \lambda\in\Lambda}}(D_j\cap\overline{D}_\lambda)\} =$$

$$= \sum_{i=1}^{L} B_i(t,T_i)[1-(1-\Delta_i)E\{\chi(\omega,D_i)\mid \bigcap_{\substack{j\in J\\ \lambda\in\Lambda}}(D_j\cap\overline{D}_\lambda)\}] =$$

$$= \sum_{i=1}^{L} B_i(t,T_i)[1-(1-\Delta_i)\frac{P\{D_i\cap\bigcap_{\substack{j\in J\\ \lambda\in\Lambda}}(D_j\cap\overline{D}_\lambda)\}}{P\{\bigcap_{\substack{j\in J\\ \lambda\in\Lambda}}(D_j\cap\overline{D}_\lambda)\}}]$$

Then the future expected value of the portfolio immediately after T_1

$$E\{\Pi(T_p+0,T;\omega)\mid \bigcap_{\substack{j\in J\\ \lambda\in\Lambda}}(D_j\cap\overline{D}_\lambda)\} =$$

$$= \sum_{i=p}^{L} n_i E\{R_i(T_p+0,T_i;\omega)\mid \bigcap_{\substack{j\in J\\ \lambda\in\Lambda}}(D_j\cap\overline{D}_\lambda)\} =$$

$$= \sum_{i=p}^{L} n_i B(T_p,T_i)[1-(1-\Delta_i)\frac{P\{D_i\cap\bigcap_{\substack{j\in J\\ \lambda\in\Lambda}}(D_j\cap\overline{D}_\lambda)\}}{P\{\bigcap_{\substack{j\in J\\ \lambda\in\Lambda}}(D_j\cap\overline{D}_\lambda)\}}]$$

Thus, assuming that portfolio’s securities defaults only at their maturities we could present the loss distribution of the portfolio in the compact form. An important risk characteristic of the multi name corporate securities is the loss distribution. Let us introduce a corporate portfolio in which number of shares is the same $n_i = n$ for all securities An example of such portfolio is a corporate equity or debt indexes. The portfolio loss is a cash flow which can be represented by a random stepwise function that can be written in the form

$$L_\Pi(t, \omega) = n \sum_{j : T_j \le t} \chi(D_j)$$

Recall that we assumed that the only moments when the function $L_\Pi(t, \omega)$ can change its value are maturity dates. Therefore, in order to calculate statistical characteristics of the portfolio losses one needs to define the joint distribution of the random variables $L_i(\omega) = L_\Pi(T_i, \omega)$, $i = 1, \ldots, N$. Let q, $q < k + 1$ be an integer and $t \in [T_k, T_{k+1})$. Then

$$P\{ L_\Pi(t, \omega) = q \} = P\{ L_\Pi(T_{k-1}, \omega) = q \} = \tag{6}$$

$$= \binom{k-1}{q} E \sum_{Q} \prod_{j \in Q} \prod_{\lambda \in K \setminus Q} \chi_j \bar{\chi}_\lambda = \binom{k-1}{q} \sum_{Q} P\{ \bigcap_{\substack{j \in Q \\ \lambda \in K \setminus Q}} D_j \cap \bar{D}_\lambda \}$$

where Q is a finite set of the integers chosen from the set $K = \{1, 2, \ldots, k\}$. The probabilities on the right hand side of the above formula (6) can be calculated by using formula (5). We have presented the portfolio loss distribution assuming that defaults might occur at maturity of the bonds.

The default time is another characteristic of the risky portfolio. By using default time one can represent new valuation approach for the pricing problem. Suppose that default of the bonds R_i can occurs at the dates T_j, $j = 1, 2, \ldots i$. Let τ denote the time of the first default of the portfolio (6). Then τ is a random variable taking values $T_1, .. T_L$. The value $+\infty$ we assign for $\tau(\omega)$ for no default scenario ω during the lifetime of the portfolio. Next table represents the distribution of the first default time assuming that τ has the non homogeneous binomial distribution

Table 1

τ	T_1	T_2	....	T_L	$+\infty$
distribution	P_1	$Q_1 P_2$	...	$Q_1 \ldots Q_{L-1} P_L$	$Q_1 \ldots Q_{L-1} Q_L$

Here P_i denotes the probability of default at the moment T_i and $Q_i = 1 - P_i$.

The value of the portfolio at the moment of the first default can be presented in the form

$$\Pi(\tau, \overline{\mathbf{T}}; \omega)\,\chi(\tau \le T_L) = \sum_{j=1}^{L} \Pi(T_j, \overline{\mathbf{T}}; \omega)\,\chi(\tau = T_j) =$$

$$= \sum_{j=1}^{L} \sum_{i=j}^{L} n_i R_i(T_j, T_i; \omega)\,\chi(\tau = T_j) = \sum_{j=1}^{L} \chi(\tau = T_j) \sum_{i=j}^{L} n_i \Delta_i B(T_j, T_i)$$

Accepting a hypothesis regarding the default time distribution one can easy calculate risk characteristics of the portfolio. Introduce the random moment τ_k which denotes the k-th , consecutive default k = 1, 2, ... of the portfolio. Then the distribution of the k-th default can be presented in explicit form. In order to define distribution of the default we need to calculate the distribution $P\{\tau_k = T_j\}$ when j = 1, 2, …, L . For example, if k = 2 then we note that

$P\{\tau_2 = T_1\} = 0$, and

$P\{\tau_2 = T_2\} = P\{\tau_1 = T_1, \tau_2 = T_2\} = P_{12}$

From formula (2) it follows that

$$P\{\tau_2 = T_k\} = \sum_{j=1}^{k-1} P\{\tau_1 = T_j, \tau_2 = T_k\} = \sum_{j=1}^{k-1} E\chi\{\bigcap_{i=1}^{j-1} \overline{D}_i\}\,\chi\{D_j\}\,\chi\{\bigcap_{h=j+1}^{k-1} \overline{D}_h\} \times$$

$$\times\ \chi\{D_k\} = \sum_{j=1}^{k-1} E\,\frac{\theta_j}{1-\Delta_j} \prod_{i=1}^{j-1}\left(1 - \frac{\theta_i}{1-\Delta_i}\right) \times \frac{\theta_k}{1-\Delta_k} \prod_{h=j+1}^{k-1}\left(1 - \frac{\theta_h}{1-\Delta_h}\right)$$

k = 2, 3, ..., L. Recall that $P\{\tau_1 = T_j, \tau_2 = T_k\}$ is the probability that the first credit event is the default of the j-th bond at its maturity T_j , j = 1, 2, … k – 1 and the second credit event is the default of the k-th bond at T_k . For arbitrary k > 2 the default distribution $P\{\tau_k = T_j\}$, j = k, k + 1, …, L can be represented by the formula

$$P\{\tau_k = T_j\} = \sum_{J_{k-1}} P\{\tau_1 = T_{j1}, \tau_2 = T_{j2}, \dots, \tau_{k-1} = T_{j\,k-1}, \tau_k = T_j\} =$$

$$= E\chi\{D_j\} \sum_{J_{k-1}} \prod_{i \in J_{k-1}} \chi\{\overline{D}_i\} \prod_{\lambda \in J \setminus J_{k-1}} \chi\{D_\lambda\} = E\,\frac{\theta_j}{1-\Delta_j} \sum_{J_{k-1}} \prod_{i \in J_{k-1}} \frac{\theta_i}{1-\Delta_i} \prod_{\lambda \in J\setminus J_{k-1}} \left(1 - \frac{\theta_\lambda}{1-\Delta_\lambda}\right) \qquad (7)$$

where the set $J_{k-1} = \{ j1, j2, \dots j\,k-1\}$ is a subset of $J = \{1, 2, \dots, j\}$, $1 \le k \le j \le L$.

Let us consider a more general situation. The structure we will represent below is close to CDS basket trenching. Assume that the bond with maturity T_i might default also at the prior maturity dates T_j , $1 \le j \le i \le L$. Taking into account equality

$$\chi(\tau_i > T_L) = 1 - \sum_{j=1}^{L} \chi(\tau_i = T_j)$$

the i-th company corporate bond price (1) can be presented in the form

$$R_i(t, T_i; \omega) = \sum_{j=1}^{i} \Delta_i B(t, T_j)\, \chi(\tau_i(\omega) = T_j) + B(t, T_i)\, \chi(\tau_i(\omega) > T_i) =$$

$$= B(t, T_i) - \sum_{j=1}^{i} [\, B(t, T_i) - \Delta_i B(t, T_j)\,]\, \chi(\tau_i(\omega) = T_j) \tag{8}$$

Multiplying the first line of the formula (8) by $\{ \omega : \tau_i = T_j \}$ $i \le j$, $i = 1, \dots, L$ we note that

$$R_i(t, T_i; \omega) = \Delta_i B(t, T_j) = R_i(t, T_j; \omega)$$

Thus, in this model recovery rate does not depend on time of default. On the other hand if $\omega \in \{ \tau_i > T_i \}$ then $R_i(t, T_i; \omega) = B(t, T_i)$. Consider portfolio of risky bonds $\Pi(t, \overline{\mathbf{T}}; \omega) = \sum_{j=1}^{L} n_i R_i(t, T_i; \omega)$. Then

$$\Pi(t, \overline{\mathbf{T}}; \omega) = \Pi_0(t, \overline{\mathbf{T}}) - \sum_{i=1}^{L} \sum_{j=1}^{i} n_i [B(t, T_i) - \Delta_i B(t, T_j)]\, \chi(\tau_i = T_j) =$$

$$= \Pi_0(t, \overline{\mathbf{T}}) - \sum_{j=1}^{L} \sum_{i=j}^{L} n_i [B(t, T_i) - \Delta_i B(t, T_j)]\, \chi(\tau_i = T_j) \tag{9}$$

where

$$\Pi_0(t, \overline{\mathbf{T}}) = \sum_{i=1}^{L} n_i B(t, T_i), \quad \sum_{i=1}^{L} n_i = N.$$

The right hand side of the formula (9) represents a valuation formula of the portfolio. It shows the spread between the 0-risk portfolio and the correspondent corporate portfolio. Inequality of these two portfolios is caused by possibilities of default risky assets. For example, for the scenario $\omega \in \{ \tau_i = T_j \}$, $j = 1, 2, .. L$, $i \le j$ the difference in value between Treasury and correspondent risky portfolios at t also called spread is

$$\Pi_0(t, \overline{\mathbf{T}}) - \Pi(t, \overline{\mathbf{T}}; \omega) = \sum_{i=j}^{L} n_i [\, B(t, T_i) - \Delta_i B(t, T_j)\,]$$

Here, the loss we interpret with respect to correspondent 0-default portfolio. Let 'l' be a positive number $0 < l < 1$. Then a probability that the total loss of the corporate portfolio does not exceed level l prior of the date T_k is equal to

$$P\{ \sum_{i=1}^{L} \frac{n_i}{N} [B(t,T_i) - \Delta_i B(t,T_1)] \chi(\tau_i = T_k) < l \} \tag{10}$$

where $N = \sum_{i=1}^{L} n_i$. The formula (10) coincides with the probability that the lifetime of the equity tranche of the portfolio will exceed T_k. The probability that the equity tranche will be exhausted within the interval $(T_k, T_{k+1}]$, $k = 1, \dots, L-1$ by definition is equal to

$$P\{ \sum_{j=1}^{k} \sum_{i=j}^{L} \frac{n_i}{N} [B(t,T_i) - \Delta_i B(t,T_j)] \chi(\tau_i = T_j) \le l , \sum_{j=1}^{k+1} \sum_{i=j}^{L} \frac{n_i}{N} [B(t,T_i) -$$

$$- \Delta_i B(t,T_j)] \chi(\tau_i = T_{j+1}) > l \}$$

Let $l_1 < l_2 < \dots < l_L = 1$ be a sequence of numbers and define random moments $\lambda^{(p)}(\omega)$, $p = 1, 2, \dots, L$ putting

$$\lambda^{(p)}(\omega) = \inf\{ T_k : \sum_{j=1}^{k+1} \sum_{i=j}^{L} \frac{n_i}{N} [B(t,T_i) - \Delta_i B(t,T_j)] \chi(\tau_i = T_j) > l_p \} \tag{11}$$

and $\lambda^{(p)}(\omega) = T_L$ if the expression in braces does not exceed the barrier l_p. Then the lifetime of the tranche defined by attachment point l_{k-1} and detachment point l_k is the random semi-interval $[\lambda^{(k-1)}(\omega), \lambda^{(k)}(\omega))$. We put $t = T_0 \ge 0$ and call the $[l_{k-1}, l_k]$ tranche as the k-tranche. Let us calculate the market value of the spread of the k-tranche. This value will depend on a moment of calculation and it represents the fixed premium paid over the lifetime of the tranche in exchange for the insurance that reimburses the tranche losses. The market value of the k-trench can be represented as the difference of two equity tranches with detachment points l_k and l_{k-1} correspondingly. Consider first the $[0, l_k]$ equity tranche. The lifetime of this tranche is $[t, \lambda^{(k)}(\omega))$. The cash flow received by protection buyer from protection seller is equal to the tranche losses. On the other hand protection buyer pays a periodic coupon $s_k^{(j)}$ over the lifetime of the tranche. For example, suppose that for some ω, $T_j \in [\lambda^{(k-1)}(\omega), \lambda^{(k)}(\omega))$. Then for this scenario applying PV reduction for the tranche valuation we arrive at the equality

$$\sum_{q=1}^{T_q \le T_j} \sum_{i=k}^{L} \frac{n_i}{N} (1 - \Delta_i) B(T_0,T_q) \chi(\tau_i = T_q) = s_k^{(j)} \sum_{u=1}^{j} B(T_0,T_u)$$

In general case we have

$$\sum_{j=1}^{L} \chi(\lambda^{(k)} = T_j) \sum_{q=1}^{T_q \le T_j} \sum_{i=k}^{L} \frac{n_i}{N} (1 - \Delta_i) B(T_0, T_q) \chi(\tau_i = T_q) =$$

$$= \sum_{j=1}^{L} \chi(\lambda^{(k)} = T_j)\, s_k^{(j)}(\omega) \sum_{u=1}^{j} B(T_0, T_u)$$

From this formula it follows that the market spread depends on the lifetime of the tranche. As far as this lifetime is random it is also an additional risk factor that should be taking into account for the pricing of the risk. Putting

$$S_k^{(\lambda)}(\omega) = \sum_{j=1}^{L} \chi(\lambda^{(k)} = T_j)\, s_k^{(j)}(\omega)$$

we note that the solution of the equation is

$$S_k^{(\lambda)}(\omega) = \sum_{j=1}^{L} \chi(\lambda^{(k)} = T_j) \frac{\sum_{q=1}^{T_q \le T_j} \sum_{i=k}^{L} \frac{n_i}{N} (1 - \Delta_i) B(T_0, T_q) \chi(\tau_i = T_q)}{\sum_{u=1}^{j} B(T_0, T_u)}$$

is the market spread value of the k-tranche that depends on scenario. On the other hand spot price used for trades of the tranche implies the market risk.

Consider the basket insurance problem with respect to outstanding value. The losses of the portfolio over $[T_0, T_u]$, u = 1, 2, … L is a stepwise random function in u

$$\Theta(u, \omega) = \sum_{j=1}^{u} \sum_{i=j}^{L} \frac{n_i}{N} (1 - \Delta_i) \chi(\tau_i = T_j) \quad , \quad u \in \overline{\mathbf{T}}$$

Let us introduce the lifetime of the $[0, l_k]$-tranche. Denote

$$\theta^{(k)}(\omega) = \inf\{ u \in \overline{\mathbf{T}} : \Theta(u, \omega) > l_k \} \tag{12}$$

k = 1, 2, …, L . The sequence $\boldsymbol{\theta} = \{ \theta^{(k)}(\omega), k = 1, 2, \dots, L \}$ of the random times can be used to define future value, FV of the cash flow generated by the losses. To determine the full protection of the $\boldsymbol{\theta}$-tranche $[l_{k-1}, l_k]$ we first find the spread for the scenarios when the tranche's default time is equal to T_j . Indeed, the spread s_k which would be periodically paid up to $\theta^{(k)}$ to a protection seller in exchange for the full protection can be defined as following. Note that for the scenario $\{ \omega : \theta^{(k)}(\omega) = T_j \}$ the date -T_j FV of the k-equity tranche is defined as

$$\sum_{q=1}^{j} \sum_{i=j}^{L} \frac{n_i}{N} (1 - \Delta_i) B^{-1}(T_q, T_j) \chi(\tau_i = T_q) = s_k^{(j)} \sum_{q=1}^{j} B^{-1}(T_q, T_j)$$

The left hand side of the equality represents losses of the portfolio corporate bonds having maturities $T_j, \dots, T_L$ that default at the date T_j. On the other hand if l_k-tranche occurs at T_j then the protection payment would be paid j times to the protection seller prior to T_j. Then the solution of the latter equation is

$$s_k^{(j)} = \frac{\sum_{q=1}^{j} \sum_{i=j}^{L} \frac{n_i}{N} (1 - \Delta_i) B^{-1}(T_q, T_j) \chi(\tau_i = T_q) \chi(\theta^{(k)} = T_j)}{\sum_{q=1}^{j} B^{-1}(T_q, T_j) \chi(\theta^{(k)} = T_j)}$$

Then the market spread of the k-equity tranche is equal to

$$S_k^{(\theta)}(\omega) = \sum_{j=1}^{L} s_k^{(j)}(\omega) \chi(\theta^{(k)} = T_j)$$

Then the market spread value of the $[\, l_{k-1}, l_k \,]$-trench is $S_k(\omega) - S_{k-1}(\omega)$ representation. The spot market price of the trench at the date t is formed by the market supply-demand relationship at t.

CDOs valuation.

A CDO is a type of structured asset-backed security (ABS) presented in the form of a trenchant portfolio. Each tranche is a bilateral contract between protection sellers and protection buyers. The protection seller receives a fixed coupon (spread) of the outstanding notional of the tranche from the tranche buyers. The CDO buyer is a buyer of the protection. From the time when the total portfolio loss crosses the attachment point of the tranche the protection seller pays for any loss of the tranche. Let t and $t_1 < \dots < t_L$ be a current moment of time and payment dates correspondingly. If the time t is the first payment date then the CDO is called funded or partially funded. Otherwise the tranche is unfunded. Assume that the trench is unfunded and therefore the first protection payment is scheduled on the date t_1. The valuation of the unfunded tranche could be easily adjusted to cover the funded CDO by accepting that the first payment at t.

Remark. There are several basic distinctions between our approach and the others commonly used benchmarks. The first difference is that in order to value the CDO tranche the ‘expected present values’ or the ‘risk-neutral’ expected present value are used to write equality of the two legs of the tranche. As we highlighted earlier in [3] this reduction eliminates market risk which is implied by the market price of the derivatives. This approach in derivative pricing remarkably oversimplifies valuation by reducing real theoretical risk. In contrast to the benchmark valuation we present valuation of the transactions for each scenario. In this case we deal with the market price in the sense that each scenario implies a particular price of the instrument. Thus, the market prices of the tranche as well as its lifetime are random variables. The stochastic nature of the lifetime of a tranche could not be eliminated or ignored. Missing this factor leads to the loss of the general risk in tranche pricing. On the other hand we also need to remark that expected value of the market spread does not coincide with the value of the tranche which is derived by using expected values of the cash flows represented the tranche seller and buyer.

Denote $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. In a discrete scheme for the writing simplicity we relate losses occurred during a period $(t_{j-1}, t_j]$ to the date t_j. Then the PV of the cash flow from protection seller to protection buyer is the loss of the portfolio occurred during the period $[t_u \wedge \theta^{(k-1)}, t_u \wedge \theta^{(k)}]$. The right hand side (9) represents total portfolio losses. Its reduction over the period $[t_u \wedge \theta^{(k-1)}, t_u \wedge \theta^{(k)}]$ leads to

$$\Lambda(t_u \wedge \theta^{(k)}; \omega) = \sum_{t_j = t_u \wedge \theta^{(k-1)}}^{t_u \wedge \theta^{(k)}} \sum_{i=j}^{L} n_i (1 - \Delta_i) B(t, t_j) \chi(\tau_i = t_j)$$

Recall that $\Delta_i \geq 0$ is the recovery rate and $n_i(1 - \Delta_i)$ is the loss of the i-th asset at the default event at τ_i. The lifetime of the k- tranche is the random time interval $[t \vee \theta^{(k-1)}, T_L \wedge \theta^{(k)}]$ where the random variables $\theta^{(j)}$ $j = 1, \dots$ are defined in (12). The moment $\theta^{(k)}$ is simultaneously the last moment of the lifetime of the k-tranche and the starting moment of the next (k + 1)-tranche and the value of the outstanding principal of the tranche specifies the premium related to the exhausting and on-the-run tranche. The protection buyer pays a spread s_k upon the notional outstanding at the scheduled dates $T_1, \dots, T_N = T$ which fall into the time interval. Putting $\Delta T = T_j - T_{j-1}$ we note that the number of protection payments during the lifetime of the k-tranche is a random variable

$$\frac{T \wedge \theta^{(k)} - t \vee \theta^{(k-1)}}{\Delta T}$$

Thus, if for particular market scenario a period $(T_{u-1}, T_u]$ belongs to the lifetime of the k-tranche then for this ω the outstanding amount at next premium date T_u is

$$N l_k - \sum_{j : t_j \in [T_{u-1}, T_u]} \sum_{i=j}^{L} n_i \Delta_i \chi(\tau_i = t_j)$$

Multiplying the PV of this outstanding by s_k and then summing up over all appropriate u we arrive at the PV of the premium leg of the k-tranche. The equality of two legs of the k-tranche leads to the equation

$$\sum_{t_j = t_1 \vee \theta^{(k-1)}}^{t_L \wedge \theta^{(k)}} \sum_{i=j}^{L} n_i (1 - \Delta_i) B(t, t_j) \chi(\tau_i = t_j) =$$

$$= s_k \sum_{T_u = T_1 \vee \theta^{(k)}}^{T_N \wedge \theta^{(k+1)}} B(t, T_u) [N l_k - \sum_{j : t_j \in [T_{u-1}, T_u]} \sum_{i=j}^{L} n_i \Delta_i \chi(\tau_i = t_j)] \tag{13}$$

Expression in the brackets on the right hand side of (13) is the outstanding remainder of the principal at T_u. From the equation (13) it follows that the spread is a random variable

$$s_k(\omega) = \frac{\sum_{t_j = t_1 \vee \theta^{(k-1)}}^{t_L \wedge \theta^{(k)}} \sum_{i=j}^{L} n_i (1 - \Delta_i) B(t, t_j) \chi(\tau_i = t_j)}{\sum_{T_u = T_1 \vee \theta^{(k)}}^{T_N \wedge \theta^{(k+1)}} B(t, T_u)[N l_k - \sum_{j : t_j \in [T_{u-1}, T_u]} \sum_{i=j}^{L} n_i \Delta_i \chi(\tau_i = t_j)]} \tag{14}$$

Formula (14) represents the market value of the spread derived from the equality of the present values of the two legs of the k-tranche. This is the exact solution presents market value of the k-tranche. Admitting a hypothetical default distribution of the assets one can attempt to calculate expected value and higher moments of the market spread and calculate the risk characteristics implied by the spot price. Indeed, let $< s_k(t) >$ denote a spot value of the spread at t. If the tranche spread increases then prospective losses and probability of default will also increase. The buyer pays less for more risky assets while the tranche seller receives less than needed to cover risky assets. Conversely, if the spot tranche spread narrowing then buyer pays higher price for protection and the chance to default of the underlying securities becomes considerably lower.
Remark. Recall that the well-known reduced form approach for valuation CDO uses risk neutral expectation to reduce real cash flows. As it showed above that there is no necessity for risk neutral problem setting.

Comments.
Here we comment some benchmark approaches which commonly used for pricing and hedging risky portfolios. We briefly outline bellow the copula and perfect copula constructions and its application to joint default modeling. Then we present some critical comments.

I. Let us consider a portfolio of risky arbitrary structured securities. This is a basket of correlated corporate securities. One of the most popular approaches has been used for valuation a risky multi name structures is a copula approach. It was introduced in the Gaussian form in [5]. Later different copulas were introduced to cover statistical differences between historical data and Gaussian hypothetical distribution [1]. Let us briefly recall primary results. For simplicity we suppose that the all functions used next are continuous. Existing of the copula is represented by the next theorem.

Theorem. Let $F(x)$ be a distribution function in n-dimensional Euclid space and $F_1, \ldots, F_n$ be the set of the one-dimensional marginal distribution functions constructed from F. Then there exist a distribution function $C(x)$ on $[0, 1]^n$ such that

$$F(x_1, \ldots x_n) = C(F_1(x_1), \ldots F_n(x_n)) \tag{15}$$

for an arbitrary $x \in [0, 1]^n$.

Note that the copulas C as well as the one dimensional marginal distributions F_j on the right hand (15) are uniquely defined by the known distribution function F.

In application when a set of marginal distributions $f_i(x_i)$ are given and a particular function $g(x)$ is claimed to be a copula we should be aware that

$$G(x) = g(f_1(x_1), \ldots f_n(x_n))$$

is multidimensional distribution function, i.e. for example that $G(x)$ is an increasing in each x_j and $0 \leq G(x) \leq 1$. Otherwise copula and marginal distributions do not have the same multidimensional distribution function. The exceptions might be the case when marginal distributions are uniquely defines the joint distribution.

In reduced form approach default is implied by the prices of corporate bonds. In this case copula approach is applied to the time of defaults which are interpreted as random variables. In structural approach two types of assets are involved. These are stocks and bonds of a company. Follow R. Merton [6] default of the company occurred when the stock price reaches a particular barrier specified by the company debt. In contrast to risk-neutral probability Q that commonly used in research papers which call for heuristic replacing real assets on its risk-neutral virtual counterparts we use a complete probability space $\{\Omega, F, P\}$ that in finance is associated with the 'real' world. In Black-Scholes theory they derived option pricing formula in which real underlying was replaced by the virtual underlying having risk neutral return. Later, the risk neutralization has been extended for other instruments. Note that if one wish to use risk neutral probability space it can be done by first defining the securities processes on the 'real' probability space where they initially were defined regardless whether we are going to study its derivatives and then perform the change measure transformation

$$Q(A) = \int_A g(\omega)\, dP(\omega)$$

where the density $g(\omega)$ can be defined explicitly by using Girsanov measure change techniques. Nevertheless, we need to note that in this case the calculations of the expected value of the cash flows on the use risk neutral probability space will revert the original parameters the hidden in the density g .

To present conditional independence of the times of default we need to perform calculation of the survival joint distribution. One factor models assume that for a given sequence of the Gaussian ρ-correlated random variables ξ_j the next formula takes place

$$\xi_j = \rho \varepsilon + \sqrt{1-\rho^2}\, Z_j \tag{16}$$

where ε and Z_j are mutually independent Gaussian distributed random variables. First note that presentation (16) was first applied for credit derivatives in [1]. Let us recall this result.

Statement. Let $\xi_j = \xi_j(t)$ be ρ-correlated Wiener processes

$$E[\Delta \xi_j(t)]^2 = \Delta t, \quad E\, \Delta \xi_i(t) \Delta \xi_j(t) = \rho \Delta t, \quad i \neq j$$

where $\Delta \xi_j(t) = \xi_j(t + \Delta t) - \xi_j(t)$, $j = 1, 2, \dots, n$. Then the Wiener processes $\xi_j(t)$ admit representation in the form (16).

We represent a simple generalization of the similar statement used above for the proof of the formula (0.3). Let $U(t)$ be a Wiener process independent on the given Wiener processes $\xi_j(t)$, $j = 1, 2, \dots, n$. Putting

$$\varepsilon(t) = a \sum_{i=1}^{n} \xi_i(t) + bU(t), \qquad Z_i(t) = \frac{1}{\sqrt{1-\rho}} \left(\xi_i(t) - \varepsilon(t)\sqrt{\rho} \right)$$

where

$$a = \frac{\sqrt{\rho}}{1 + (n-1)\rho} \quad \text{and} \quad b = \frac{\sqrt{1-\rho}}{\sqrt{1 + (n-1)\rho}}$$

we can easy to check that $\varepsilon(t)$ and $Z_i(t)$, $i = 1,2,\dots n$ are independent Wiener processes. Though the decomposition (16) is mathematically correct one could probably note that this decomposition is conditional in sense that if the Wiener process $U(t)$ does not exist then (16) does not hold. As far as this composition is used for the portfolio when n tends to infinity one need to define explicitly the 'risk' factor $U(t)$ which should be independent on infinite set of Wiener set $\xi_j(t)$. Otherwise we could not use the decomposition (16) as well as the limit asymptotic that follows from the decomposition. The equality (16) is used for the representation of the individual default times. Let $\zeta(\omega)$ be arbitrary random variable having a continuous in x cumulative distribution function (cdf) $G(x) = P\{\zeta(\omega) < x\}$. Consider random variable $G(\zeta(\omega))$. Then

$$P\{G(\zeta(\omega)) < t\} = P\{\zeta(\omega) < G^{-1}(t)\} = G(G^{-1}(t)) = t$$

This equality proves that the random variable $G(\zeta(\omega))$ has uniform distribution on $(0, 1)$ regardless of distribution of the random variable $\zeta(\omega)$.

For arbitrary random variables τ and ξ with cumulative distribution functions F and Φ correspondingly we have

$$P\{F(\tau(\omega)) < t\} = P\{\Phi(\xi(\omega)) < t\} = t$$

Specific interpretation of this fact was put as the initial step for the application to credit derivatives pricing. Assume that τ_j is the time of default of the i-security. In [5] the default times were assumed to be written in the form

$$\tau_j = F_j^{-1} (\Phi (\xi_j)) \qquad \text{(L)}$$

j = 1, 2, …, n. Here F_j^{-1} is the inverse function to the marginal cumulative distribution function F_j of the time of default , Φ is the standard one dimensional Gaussian cdf, and ξ_j are independent standard Gaussian variables j = 1, 2, …, n. Hence, the equality (L) shows the connection of the observable default times τ_j and the auxiliary correlated set of Gaussian random variables ξ_j .

Comment. Note that from the equality of the distributions it does not follow in general equality of the correspondent random variables on original probability space. Let us consider an illustrative example. Let ζ_ρ and η_ρ be two ρ - correlated random variables with equal cdfs G (x) and $E \zeta_\rho \eta_\rho = \rho$. Let ζ , η be another pair of independent random variables having the same cdfs G (x). Suppose that the equality (L) is true. Then with probability 1 it follows that

$$\zeta_\rho = G^{-1} (G (\zeta)) = \zeta \, , \quad \eta_\rho = G^{-1} (G (\eta)) = \eta$$

If the latter equalities are true then with the probability 1 then

$$P \{ \zeta_\rho < u , \eta_\rho < v \} = P \{ \zeta < u , \eta < v \} = P \{ \zeta < u \} P \{ \eta < v \}$$

This equality is incorrect and therefore we prove that dealing with marginal distributions their correlation is ignored.

Let us consider conditional surviving probabilities which are used in Gaussian copula. It could be written as

$$P\{\tau_j \le t_j \,|\, \varepsilon\} = \Phi \{ \frac{-\rho \varepsilon + \Phi^{-1}(F_j(t_j))}{\sqrt{1 - \rho^2}} \} \qquad (17)$$

Then taking into account (17) the joint distribution of the default times is the expected value of the product conditional probabilities

$$S(t_1, t_2, \ldots, t_n) = E \prod_{j=1}^{n} P\{\tau_j \le t_j \,|\, \varepsilon \} \qquad (18)$$

Let function C (x_1 , … , x_n) be defined by (15). For example in a simple case when asset prices are mutually independent the default times are also independent and from (15) it follows that

$$F(x_1, \dots x_n) = C(F_1(x_1), \dots F_n(x_n)) = \prod_{j=1}^{n} F_j(x_j)$$

Hence,

$$C(x_1, \dots, x_n) = \prod_{j=1}^{n} x_j .$$

In the case when default times admit presentation (16) copula function has more complex presentation. Indeed, taking into account formulas (18) one sees that

$$S(t_1, t_2, \dots t_n) = E \prod_{j=1}^{n} P\{\tau_j \le t_j \mid \varepsilon\} =$$

$$= \int_{-\infty}^{+\infty} d\Phi(\lambda) \prod_{j=1}^{n} \Phi\left\{ \frac{-\rho\lambda + \Phi^{-1}(F_j(t_j))}{\sqrt{1-\rho^2}} \right\}$$

Here $d\Phi(\lambda)$ denotes integration with respect to the standard Gaussian distribution related to the factor ε. From this formula follows that copula represented as

$$C(x_1, \dots, x_n) = \int_{-\infty}^{+\infty} d\Phi(\lambda) \prod_{j=1}^{n} \Phi\left\{ \frac{\rho\lambda + \Phi^{-1}(x_j)}{\sqrt{1-\rho^2}} \right\}$$

Comment. Assume that equality $P\{F(\tau_j) < t\} = P\{\Phi(\xi_j) < t\}$ is true. Then for the given correlated sequence (16) of the standard Gaussian variables ξ_j, j = 1, 2, … n we can consider conditional distribution $P\{\xi_j < t \mid \varepsilon\}$ the conditional distribution $P\{\tau_j < t \mid \varepsilon\}$ remains undefined. According to definition of the conditional distributions one needs to provide the definition of the joint distributions of two random variables ε and τ_j. Such probabilities $P\{\tau_j < t, \varepsilon < y\}$ cannot be defined as far as factor ε does not related to τ_j as far as the factor is a part of the arbitrary sequence of the random variables ξ_j.

II. In the paper [4] a generalization of the representation (16) was established. It was initially assumed that the components Z_j and the common factor ε are not Gaussian. In this case equality (16) defines the class of the random variables which we use for the correlated default time modeling. This problem differs from the above Gaussian case. Indeed, in the Gaussian case the random variables ξ_j are observable while ε and Z_j on the right hand side of (16) should be defined based on these observations. In Gaussian case under certain assumptions the decomposition (16) is proved. For the Perfect Copula as it was presented in [4] the equality (16) is set for the class of random variables that only can be used for joint default time model. The degree of reliability of this assumption remains open. If the class (16) does not sufficiently broad the solution of the problem can be fail to represent correlated default structure.

Thus, let us suppose that the common and idiosyncratic default factors are given. Recall following [4] some details of the perfect copula construction. " The perfect copula model maps x_i to t_i on a "percentile to percentile" basis. That is the 5% point on the x_i distribution is mapped to the 5% point on the t_i distribution; the 10% point on the x_i distribution is mapped to the 10% point on the t_i distribution; and so on. In general, the point $t = t_i$ is mapped to $x = x_i$ where

$$x = F_i^{-1}(Q_i (t)) \qquad \text{or equivalently} \qquad t = Q_i^{-1} (F_i (x))$$

The copula model defines a correlation structure between the t_i's while maintaining their marginal distributions. The essence of the copula model is that we do not define the correlation structure between the variables of interest directly. We map the variables of interest into other more manageable variables and define a correlation structure between those variables."

Let $\{ x_i , i = 1,2, \ldots\}$ be a class of random variables admitted presentation

$$x_i = a_i M + \sqrt{1 - a_i^2}\, Z_i \qquad (HW_1)$$

where M and the Z_i are independent random variables with mean zero and standard deviation is equal to one. In (HW_1) we chose other than in before letters for notations in order to highlight that these random variables are not Gaussian as it was assumed in (16). Let t_i , $i = 1,2, \ldots$ denote random time of default of the i-th obligator and let Q_i and F_i be the cdfs of the default time of the i-th obligator and the cdf of the random variable x_i in (HW_1) . Suppose that these cumulative distribution are continuous functions. Let u be an arbitrary number from [0 , 1]. Then the "percentile to percentile" mapping can be defined as following. Then the numbers $x = x (u , i)$ and $t = t (u , i)$ are defined such that

$$F_i (x (u , i)) = Q_i (t (u , i)) = u$$

Thus, for each ' i ' and ' u ' there exists the continuous inverse functions F_i^{-1} for which

$$x (u) = F_i^{-1}(Q_i (t (u))) \qquad (HW_2)$$

Here the sub-index ' i ' was omitted for writing simplicity. It follows from (HW_2) that

$$P \{ x_i < x (u) \,|\, M \} = H_i \left(\frac{x(u) - a_i M}{\sqrt{1 - a_i^2}} \right) \qquad (HW_3)$$

where H_i denotes the cdf of the random variable Z_i . It was stated in [4] that from the equality (HW_2) it follows also that

$$P\{t_i < t(u) \mid M\} = H_i\left(\frac{F_i^{-1}(Q_i(t(u))) - a_i M}{\sqrt{1 - a_i^2}}\right) \qquad (HW_4)$$

Here, bellow we represent some comments on above copula's construction.

Annex. Now let us present other transformation of equally ρ-correlated Wiener processes system { Z } into independent Wiener system {W } which does not presume the existence of the collateral independent factor U or M.
In case when n = 1 we have a Wiener process and the decomposition is achieved. The case n = 2 is a well-known one. Define Wiener processes $W_i(t)$, i = 1, 2 putting

$$Z_1(t) = W_1(t)$$

$$Z_2(t) = \rho W_1(t) + \sqrt{1-\rho^2}\, W_2(t)$$

From the second equality it follows that the second component of the independent system is

$$W_2(t) = \frac{Z_2(t) - \rho Z_1(t)}{\sqrt{1-\rho^2}}$$

Note that Wiener processes $W_1(t)$ and $W_2(t)$ are independent. Indeed,

$$E W_2(t) = \frac{E Z_2(t) - \rho E Z_1(t)}{\sqrt{1-\rho^2}} = 0$$

$$E[W_2(t)]^2 = \frac{1}{1-\rho^2}\{E Z_2^2(t) - 2\rho E Z_1(t) Z_2(t) + \rho^2 E Z_1^2(t)\} =$$

$$= \frac{t - 2\rho^2 t + \rho^2 t}{1-\rho^2} = t$$

$$E W_1(t) W_2(t) = E Z_1(t) \frac{Z_2(t) - \rho Z_1(t)}{\sqrt{1-\rho^2}} = \frac{E Z_1(t) Z_2(t) - \rho E Z_1^2(t)}{\sqrt{1-\rho^2}}$$

$$= \frac{\rho t - \rho t}{\sqrt{1-\rho^2}} = 0$$

Consider now the general case. Let $Z(t) = \{ Z_1(t), Z_2(t), \dots Z_n(t) \}$ be a given ρ-correlated Wiener processes system. We can start with any Wiener process from the set $Z(t)$ and let it be $Z_1(t)$. Put

$$Z_1(t) = W_1(t)$$

$$Z_j(t) = \rho W_1(t) + \sqrt{1-\rho^2}\, W_j^{(1)}(t), \quad j = 2, 3, \dots, n$$

Then it follows that

$$W_j^{(1)}(t) = \frac{Z_j(t) - \rho W_1(t)}{\sqrt{1-\rho^2}}, \quad j = 2, 3, \dots, n$$

The Wiener processes $W_j^{(1)}(t)$ are totally independent upon $W_1(t)$ but they remain correlated among themselves. Indeed,

$$E[W_j^{(1)}(t)]^2 = \frac{1}{1-\rho^2}\{ E[Z_j(t) - \rho Z_1(t)]^2 \} = t$$

$$\rho_1 = E\, W_i^{(1)}(t) W_j^{(1)}(t) = \frac{1}{1-\rho^2}\{ E[Z_j(t) - \rho Z_1(t)]^2 \} = \frac{\rho}{1+\rho}$$

for $i \neq j$ and $i, j = 2, 3, \dots, n$. Note that correlation $\rho_1 < \rho$. Thus, we arrive at the new Wiener system

$$W^{(1)}(t) = \{ W_2^{(1)}(t), \dots, W_n^{(1)}(t) \}$$

of the size $n - 1$ with equal joint correlation ρ_1. This Wiener system $W^{(1)}$ is independent upon $W_1(t) = Z_1(t)$. Now we can repeat transformations that were initially applied for original system $Z(t)$ taking into account that its size is now $n - 1$. We put

$$W_2^{(1)}(t) = W_2(t)$$

$$W_j^{(1)}(t) = \rho_1 W_2(t) + \sqrt{1-\rho_1^2}\, W_j^{(2)}(t) \qquad j = 3, 4, \dots, n$$

Then similarly to above we can justify that the $W^{(2)}$ is the Wiener system

$$W^{(2)}(t) = \{ W_3^{(2)}(t), \dots, W_n^{(2)}(t) \}$$

of the size n – 2 is independent upon Wiener processes $W_2(t)$ and $W_1(t)$ and has a joint correlation ρ_2

$$\rho_2 = \frac{\rho_1}{1 + \rho_1} = \frac{\rho}{1 + 2\rho}$$

One can easy remark that (k – 1) -th step leads us to the system the Wiener processes

$$W^{(k-1)}(t) = \{ W_k^{(k-1)}(t), \ldots, W_n^{(k-1)}(t) \}$$

having equal joint correlation

$$\rho_k = \frac{\rho_{k-1}}{1 + \rho_{k-1}}$$

and it is independent on the system

$$W_{(k-1)}(t) = \{ W_1(t), \ldots, W_{k-1}(t) \}$$

Using mathematical induction we can prove that for arbitrary k

$$\rho_k = \frac{\rho}{1 + k\rho}$$

Indeed, assuming that equality holds for k – 1 we get

$$\rho_k = \frac{\rho_{k-1}}{1 + \rho_{k-1}} = \frac{\dfrac{\rho}{1 + (k-1)\rho}}{1 + \dfrac{\rho}{1 + (k-1)\rho}} = \frac{\rho}{1 + k\rho}$$

The joint correlation formula has proved for any finite number k. On the last step we have

$$W_{n-1}^{(n-2)}(t) = W_{n-1}(t)$$

$$W_n^{(n-2)}(t) = \rho_{n-2} W_{n-1}(t) + \sqrt{1 - \rho_{n-2}^2}\, W_n^{(n-1)}(t)$$

$$W_n^{(n-1)}(t) = W_n(t)$$

where the Wiener process $W_n(t)$ is independent on $W_{n-1}(t)$. Thus, applying a special form of the linear transformations to the equally correlated system $Z(t) = \{Z_1(t), \ldots, Z_n(t)\}$ arrive at the independent Wiener processes system $W(t) = \{W_1(t), \ldots, W_n(t)\}$. Now it is not difficult to present the closed form of these transformations. Indeed,

$$Z_1(t) = W_1(t) ,$$

$$Z_2(t) = \rho_0 W_1(t) + \sqrt{1-\rho_0^2}\, W_2(t) ,$$

$$Z_3(t) = \rho_0 W_1(t) + \rho_1 \sqrt{1-\rho_0^2}\, W_2(t) + \sqrt{1-\rho_1^2}\sqrt{1-\rho_0^2}\, W_3(t) ,$$

....

$$Z_k(t) = \rho_0 W_1(t) + \rho_1 \sqrt{1-\rho_0^2}\, W_2(t) + \rho_2 \sqrt{1-\rho_1^2}\sqrt{1-\rho_0^2}\, W_3(t) + \ldots$$

$$\ldots + \rho_{k-2} \prod_{i=0}^{k-3} \sqrt{1-\rho_i^2}\, W_{k-1}(t) + \prod_{i=0}^{k-2} \sqrt{1-\rho_i^2}\, W_k(t) =$$

$$= \rho_0 W_1(t) + \sum_{j=1}^{k-2} \rho_j \prod_{i=0}^{j-1} \sqrt{1-\rho_i^2}\, W_{j+1}(t) + \prod_{i=0}^{k-2} \sqrt{1-\rho_i^2}\, W_k(t)$$

$k = 2, 3, \ldots n$. In this representation we did not assume the existing the independent risk factor U. The problem whether or not the Vasicek's limit distribution exists without the assumption regarding the existence of the process U remains open.

References.

1. Burtschell, X., Gregory, J., Laurent, J.-P., A comparative analysis of CDO pricing models. Working Paper, 2005.

2. Gikhman, I., Corporate Debt Pricing, http://papers.ssrn.com/sol3/papers.cfm?abstract_id=10185373.

3. Gikhman, I., Risky Swaps. ICFAI Journal of Derivative Markets. July 2008.

4. Hull, J. and White, A., The Perfect Copula, Working Paper, University of Toronto, 2005.

5. Li, D., On Default Correlation: A Copula Approach, Journal of Fixed Income, 9, 2000, 43-54.

6. Merton, R., On the pricing of corporate debt: The risk structure of interest rates. Journal of Finance, 2, 449-471.

7. Vasicek, O.A., Probability of loss on loan portfolio, KMV Corporation, 1987.

8. Vasicek, O., Limiting Loan Loss Distribution, KMV Corporation, 1991.

9. Vasicek, O., The Distribution of Loan Portfolio Value, Risk, December 2002.